1. (**Gaussian Elimination and Schur Complement**)

   Let $A \in \mathbb{R}^{m\times m}$ be nonsingular. Suppose that for each k with $1 \le k \le m$, the upper-left $k \times k$ block of A is nonsingular. Assume that A is written in the block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$.

   (a) Verify the formula

   $$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

   for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schur complement* of A_{11} in A.

   (b) Suppose now that A_{21} is eliminated row by row by means of n *steps of Gaussian elimination without pivoting*:

$U = A,\ L = I$
for $k = 1$ to n
for $j = m-n+1$ to m
$l_{jk} = u_{jk}/u_{kk}$
$u_{j,k:m} = u_{j,k:m} - l_{jk}u_{k,k:m}$

   Show that the bottom-right $(m-n) \times (m-n)$ block of the result is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

2. (**Exponential Interpolation**)

   Some modeling considerations have mandated a search for a function

   $$u(x) = c_0 e^{c_1 x + c_2 x^2}$$

   where the unknown coefficients c_1 and c_2 are expected to be nonpositive. Given are data pairs to be interpolated, (x_0, z_0), (x_1, z_1), and (x_2, z_2), where $z_i > 0$, $i = 0, 1, 2$. Thus, we require $u(x_i) = z_i$.

   The function $u(x)$ is not linear in its coefficients, but $v(x) = \ln(u(x))$ is linear in its.

   Find a quadratic polynomial $v(x)$ that interpolates appropriately defined three data pairs, and then give a formula for $u(x)$ in terms of the original data.

3. The gradient method (the steepest descent method) for solving a linear system, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a real and symmetric positive definite $n \times n$ matrix, is given as follows:

Given initial guess $\mathbf{x}_0$, for $k \ge 0$, we compute
(i) $\mathbf{g}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b}$,
(ii) $\rho_k = \mathbf{g}_k^t\mathbf{g}_k / \mathbf{g}_k^t\mathbf{A}\mathbf{g}_k$,
(iii) $\mathbf{x}_{k+1} = \mathbf{x}_k - \rho_k\mathbf{g}_k$.

   (a) Verify that $\mathbf{g}_k \cdot \mathbf{g}_{k+1} = 0$ for $k = 0, 1, \ldots$.

   (b) Via the result in (a), can we prove that the gradient method always converges at most in n iterations? Justify your answer.

   (c) For the following descent method,

Given initial guess $\mathbf{x}_0$, for $k \ge 0$, we compute
(i) $\mathbf{g}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b}$,
(ii) $\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha\mathbf{g}_k$,

   find the conditions on α so that the revised descent method converges.

4. For computing the integral $\int_{-\pi/2}^{\pi/2} \cos(x) f(x)\, dx$, find a two point quadrature formula

$$S_2(f) = c_1 f(x_1) + c_2 f(x_2),$$

which is exact for all polynomials of a maximal possible degree.

5. The modified Euler method for the approximation of the Cauchy problem is defined as:

$$\boxed{\begin{array}{l} u_{n+1} = u_n + hf(t_{n+1}, u_n + hf(t_n, u_n)) \\ u_0 = y_0 \end{array}}$$

Find the region of stability for this method when applied to the test problem

$$\begin{cases} y'(t) = \lambda y(t), \quad t > 0 \\ \quad y(0) = 1, \end{cases}$$

where $\lambda \in \mathbb{R}^-$.

6. Consider the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Is it possible to find the Cholesky factorization of A? If so, find the unique upper triangular matrix H such that $A = H^T H$.

Assume that you have computed the upper triangular matrix affected by rounding errors $\widetilde{H}$ with $\widetilde{H}^T \widetilde{H} = A + \delta A$. Find an estimate of $\|\delta A\|_2$ for the given matrix A.

7. Find all the functions $f(x) = a_2 x^2 + a_1 x + a_0$ whose polynomial of best approximation of degree 1 on the interval $[2, 4]$ is $p_1^*(x) = 0$.

8. (**Interpolation and Weak Line Search**)

A popular technique arising in methods for minimizing functions in several variables involves a *weak line search*, where an approximate minimum x^* is found for a function in one variable, $f(x)$, for which the values of $f(0)$, $f'(0)$, and $f(1)$ are given. The function $f(x)$ is defined for all nonnegative x, has a continuous second derivative, and satisfies $f(0) < f(1)$ and $f'(0) < 0$. We then interpolate the given values by a quadratic polynomial and set x^* as the minimum of the interpolant.

(a) Find x^* for the values $f(0) = 1$, $f'(0) = -1$, $f(1) = 2$.

(b) Show that the quadratic interpolant has a unique minimum satisfying $0 < x^* < 1$. Can you show the same for the function f itself?

9. (**Gaussian Elimination**)

Given an m-by-m nonsingular matrix A, how do you efficiently solve the following problems, using Gaussian elimination with partial pivoting?

(a) Solve the linear system $A^k x = b$, where k is a positive integer.

(b) Compute $\alpha = c^T A^{-1} b$.

(c) Solve the matrix equation $AX = B$, where B is m-by-n.

You should: (1) describe your algorithms, (2) present them in pseudocode (using a Matlab-like language), and (3) give the required flops.

10. Let $\mathbf{A}$ be a strictly diagonally dominant $n \times n$ matrix. Show that the Jacobi iterative method generates a convergent sequence of approximate solutions when applying it to solve the linear system $\mathbf{Ax} = \mathbf{B}$ for any initial guess $\mathbf{x}_0$.

11. Find solutions of the two systems of equations:

$$\begin{cases} x_1 + 3x_2 &= 4 \\ x_1 + 3.00001x_2 &= 4.00001 \end{cases} \iff A_1 x = b.$$

and

$$\begin{cases} y_1 + 3y_2 &= 4 \\ y_1 + 2.99999y_2 &= 4.00001 \end{cases} \iff A_2 y = b.$$

Compute $\|A_1 - A_2\|_\infty$ and $\|x - y\|_\infty$. Using the notion of the matrix condition number, explain why $\|x - y\|_\infty$ is much larger than $\|A_1 - A_2\|_\infty$.

12. Consider the following two fixed point methods to find the root $z \approx 0.6$ of the equation $x + \ln x = 0$:

$$1)\ x_{n+1} = -\ln x_n, \qquad 2)\ x_{n+1} = \exp(-x_n).$$

Study the convergence of the methods and argue which one you would prefer.

13. Find the polynomial of best approximation $p_1^*(x)$ for $f(x) = |x|$ on $[-1, 3]$.

14. For the solution of the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

consider the following iterative method: given $\mathbf{x}^{(0)} \in \mathbb{R}^2$, find

$$\mathbf{x}^{(k+1)} = B(\theta)\mathbf{x}^{(k)} + \mathbf{g}(\theta) \qquad \text{for } k \geq 0,$$

where θ is a real parameter and

$$B(\theta) = \frac{1}{4}\begin{bmatrix} 2\theta^2 + 2\theta + 1 & -2\theta^2 + 2\theta + 1 \\ -2\theta^2 + 2\theta + 1 & 2\theta^2 + 2\theta + 1 \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} \frac{1}{2} - \theta \\ \frac{1}{2} - \theta \end{bmatrix}.$$

Address the following points:

(a) Check that the method is consistent $\forall \theta \in \mathbb{R}$.

(b) Determine the values of θ for which the method is convergent.

(c) Find the optimal value of θ, i.e. the value of θ for which $\rho(B(\theta))$ is minimum.

15. Consider the function $f(x) = \ln(x) + 6\sqrt{x} - 9$, which has a zero on the interval $[1, 2]$. Given two fixed point methods $x = \phi_i(x)$, $i = 1, 2$, where

$$\phi_1(x) = \frac{(9 - \ln(x))^2}{36} \qquad \text{and} \qquad \phi_2(x) = e^{9 - 6\sqrt{x}},$$

verify that the zero of f is a fixed point for ϕ_1 and ϕ_2. Which method would you use to calculate the zero of f? Justify your answer.

16. Consider $f(x) = \sin(\pi x)$ on the interval $[-1, 1]$. Find the polynomial $\Pi_2 f(x)$ interpolating $f(x)$ at the Chebyshev nodes.

17. Given a nonsingular matrix $A \in \mathbb{R}^{n \times n}$, consider the preconditioned nonstationary Richardson method:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k P^{-1}(\mathbf{b} - A\mathbf{x}^{(k)}),$$

where $P \in \mathbb{R}^{n \times n}$ is nonsingular. Find a formula for the parameter α_k such that $\mathbf{x}^{(k+1)}$ minimizes the Euclidean norm of the preconditioned residual $\|P^{-1}(A\mathbf{y} - \mathbf{b})\|_2$ among all vectors $\mathbf{y} \in \mathbb{R}^n$ of the form:

$$\mathbf{y} = \mathbf{x}^{(k)} + \alpha P^{-1}(\mathbf{b} - A\mathbf{x}^{(k)}).$$

18. (**Cholesky Factorization**)

    Given an m-by-m symmetric and positive definite matrix A, how do you efficiently solve the following problems, using the Cholesky factorization of A?

    (a) Solve the linear system $A^k x = b$, where k is a positive integer.

    (b) Compute $\alpha = c^T A^{-1} b$.

    (c) Solve the matrix equation $AX = B$, where B is m-by-n.

    You should: (1) describe your algorithms, (2) present them in pseudocode (using a Matlab-like language), and (3) give the required flops.

19. (**Orthogonal Polynomials**)

    Let $\phi_0(x), \phi_1(x), \phi_2(x), \dots$ be a sequence of orthogonal polynomials on an interval $[a, b]$ with respect to a positive weight function $w(x)$. Let $x_1, \dots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and $a < x_1 < \cdots < x_n < b$.

    (a) Show that the Lagrange polynomials of degree $n-1$ based on these points are orthogonal to each other, so we can write

    $$\int_a^b w(x) L_j(x) L_k(x)\, dx = 0, \qquad j \neq k,$$

    where

    $$L_j(x) = \prod_{k \neq j} \frac{(x - x_k)}{(x_j - x_k)}, \qquad 1 \leq j \leq n.$$

    (b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \dots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n-1$ that interpolates the function $f(x)$ at the zeros $x_1, \dots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

    $$\|p_{n-1}\|^2 = \sum_{k=1}^{n} y_k^2 \|L_k\|^2$$

    in the weighted least squares norm. This norm for any suitably integrable function $g(x)$ is defined by

    $$\|g\|^2 = \int_a^b w(x) [g(x)]^2\, dx.$$

20. Given a function $f(x) = \sin x$ on $[-\pi, \pi]$, we want to approximate f by Lagrange interpolating polynomials $P_n(x)$ with equally spaced nodes, $x_i = -\pi + \dfrac{2i\pi}{n}$, for $i = 0, 1, \dots, n$, i.e., the supporting pairs are $\{(x_i, f(x_i))\}_{i=0}^{n}$, and we have $P_n(x_i) = f(x_i)$, for $i = 0, 1, \dots, n$. Does $P_n(x)$ converge to $f(x)$ on $[-\pi, \pi]$ as $n \to \infty$?

21. We consider the initial value problem

    $$\begin{cases} y'(t) = f(t, y(t)), & t > 0 \\ y(0) = \alpha. \end{cases}$$

    Define $h > 0$ and $t_i = ih$ for $i = 0, 1, \dots$. Let w_i be the approximate solution of $y(t_i)$ obtained with the following Trapezoidal method:

    $$w_{i+1} = w_i + \frac{h}{2}(f(t_i, w_i) + f(t_{i+1}, w_{i+1})),$$

    for $i = 0, 1, \dots$, and $w_0 = \alpha$. Find the stability condition on h so that the modified Trapezoidal method is stable when applying it to the stiff problem

    $$\begin{cases} y' = -\lambda y, \\ y(0) = \alpha \end{cases}$$

    with positive λ.

22. Find a parameter $\tau \in \mathbb{R}$ such that the Richardson method

$$x^{k+1} = x^k - \tau(Ax^k - b)$$

converges to a solution of $Ax = b$, if

$$A = \begin{pmatrix} 3 & 1 & & & \mathbf{0} \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ \mathbf{0} & & & 1 & 3 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

23. Check if the following scheme approximates the equation $y' = f(x, y)$:

$$\frac{1}{2h}(y_k - y_{k-2}) = \frac{1}{2}(f(x_k, y_k) + f(x_{k-2}, y_{k-2})).$$

Find the stability region for the scheme.

24. Consider the matrix

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix}$$

where ϵ is a very small number (e.g., 10^{-15}). Prove that the matrices L and U have entries very large in absolute value.

If rounding errors are accounted for, the LU factorization yields matrices $\widehat{L}$ and $\widehat{U}$, such that $A + \delta A = \widehat{L}\widehat{U}$. Explain why for the given matrix A, we do not have control on the size of the perturbation matrix δA.

Finally, check that by using GEM with pivoting we have control on the size of δA.

25. To compute numerically the integral $I(f) = \int_0^2 f(x)\,dx$ with $f(x) = \dfrac{1}{1+x}$, consider the composite quadrature formula:

$$I_c(f) = \frac{1}{10}[f(0) + 2f(0.2) + 2f(0.4) + \cdots + 2f(1.8) + f(2)].$$

Find an estimate for:

$$|I(f) - I_c(f)|.$$

26. Let $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ be the set of Legendre polynomials satisfying the following properties:
    i. For each n, $P_n(x)$ is a polynomial of degree n.
    ii. $\int_{-1}^{1} P(x)P_n(x)\,dx = 0$ for any polynomial $P(x)$ of degree less than n.

    Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the n^{th} Legendre polynomial $P_n(x)$ and

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\,dx$$

for $i = 1, 2, \ldots, n$.

(a) Show that if $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i P(x_i).$$

(b) For $n = 2$, we have

$$\int_{-1}^{1} P(x)\,dx = P(x_1) + P(x_2),$$

for any polynomial $P(x)$ of degree at most three with $x_1 = -\sqrt{3}/3$ and $x_2 = \sqrt{3}/3$. Verify that

$$\int_{-1}^{1}\int_{-1}^{1} P(x,y)\,dx\,dy = P(x_1,x_1) + P(x_1,x_2) + P(x_2,x_1) + P(x_2,x_2)$$

where the degree of $P(x,y)$ in x (resp., y) is at most three (resp., three).

27. Let $\mathbf{A}$ be a real and symmetric positive definite $n \times n$ matrix. The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is solved by the conjugate gradient method as follows:

Given initial guess $\mathbf{x}_0$, we compute
$\mathbf{g}_0 = \mathbf{A}\mathbf{x}_0 - \mathbf{b}$, and $\mathbf{w}_0 = -\mathbf{g}_0$.
For $k \geq 0$, knowing $\mathbf{x}_k$ we compute $\mathbf{x}_{k+1}$ as follows:
(a). $\rho_k = \mathbf{g}_k^t \mathbf{g}_k / \mathbf{w}_k^t \mathbf{A} \mathbf{w}_k$,
(b). $\mathbf{x}_{k+1} = \mathbf{x}_k + \rho_k \mathbf{w}_k$,
(c). $\mathbf{g}_{k+1} = \mathbf{g}_k + \rho_k \mathbf{A} \mathbf{w}_k$,
(d). $\beta_k = \mathbf{g}_{k+1}^t \mathbf{g}_{k+1} / \mathbf{g}_k^t \mathbf{g}_k$,
(e). $\mathbf{w}_{k+1} = -\mathbf{g}_{k+1} + \beta_k \mathbf{w}_k$

Suppose that with a given initial guess $\mathbf{x}_0$, the initial error $\mathbf{x}_0 - \mathbf{x}$ has an expression of the form

$$\mathbf{x}_0 - \mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{A}^{-1}\mathbf{g}_0$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of $\mathbf{A}$. Let m be the number of nonzero coefficients in the set $\{c_1, \ldots, c_n\}$. Prove that the conjugate gradient method converges in m iterations with the initial guess $\mathbf{x}_0$.

28. To obtain an approximate solution of the initial value problem

$$y'(x) = 0, \quad y(0) = \alpha,$$

we apply the following linear multistep method:

$$\eta_{j+2} = -9\eta_{j+1} + 10\eta_j + \frac{h}{2}(13 f(x_{j+1}, \eta_{j+1}) + 9 f(x_j, \eta_j)), \quad j \geq 0.$$

Let the starting values be $\eta_0 = \alpha$ and $\eta_1 = \alpha + \epsilon$ (ϵ = machine precision). What values η_j are to be expected for arbitrary h? Is this linear multistep method convergent?

29. (**Runge-Kutta Method and Numerical Solution of ODEs**)

Consider Heun's method

$$y_{n+1} = y_n + \frac{h}{2}[f_n + f(t_{n+1}, y_n + h f_n)].$$

(a) Show that Heun's method is an explicit two-stage Runge-Kutta method.

(b) Prove that Heun's method has order 2 with respect to h.

(c) Sketch the region of absolute stability of the method in the complex plane.